Computer Algebra for Concrete Mathematics

Part 1: Introductory Examples

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April 8, 2014, JKU

Preamble

This is the first of several Mathematica notebooks supplementing the Lecture (VL) “Computer Algebra for Concrete Mathematics” - and also the respective UE.

Homework Problems (for those attending the UE)

NOTE. In the UE on Tuesday, April 29, we will go throught this notebook and discuss its content and the related CA-Homeworks.

CA Homework 1

Install the required RISC packages (see below) on your computer. To this end, go to

```
www.risc.jku.at/research/combinat/
software/source/mathematica/ergosum
```

and see the instructions there. To pick up the packages at the RISC site, you need the following access information:

```
user ID: comb
password: HaveFun
```

CA Homework 2

Consider *Mathematica*'s output to "RSolve"

In[1]:=
```
re = {-fib[k] - fib[1 + k] + fib[2 + k] == 0, fib[0] == 1, fib[1] == 1};
```

In[2]:=
```
RSolve[re, fib[k], k]
```

Out[2]= $\left\{\left\{\mathrm{fib}[k] \to \frac{1}{2}\left(\mathrm{Fibonacci}[k] + \mathrm{LucasL}[k]\right)\right\}\right\}$

Question : Is this a correct solution to the recurrence?

Some Symbolic Packages

The following packages of my RISC Combinatorics Group are used:

NOTE. First you have to pick them up at the web page given above...

In[3]:=
```
<< RISC`fastZeil`
```

Fast Zeilberger Package version 3.60
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright 1995-2009, Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[4]:=
```
<< RISC`GeneratingFunctions`
```

Package GeneratingFunctions version 0.7 written by Christian Mallinger
Copyright 1996-2009, Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Software

- Packages of the RISC Combinatorics Group
  - Freely available at: http://www.risc.jku.at/research/combinat/software
  - More than 3,000 users world-wide. (In average two password requests per week.)
  - Audience: (pure and applied) mathematicians, but also many physicists; also engineers, etc.

InputForms: Binomials

In[5]:= $\binom{\text{n\_}}{\text{k\_}}_*$:= Binomial[n, k]

In[6]:= $\binom{a}{3}_*$

Out[6]= $\frac{1}{6}(-2+a)(-1+a)\,a$

Automatic Guessing

- **I.Q. Tests**
  Setzen Sie die Reihe fort: 1 1 2 3 5 8 13 21 ?
  Es gibt zwei Lösungsmoeglichkeiten, eine leichte und eine schwierigere. Versuchen Sie, ob Sie beide finden koennen.
  [Aufgabe 13, Denksport I für Superintelligente; ``Check Your Own I.Q.'', Hans J. Eysenck, 1966]

- **Computer Solution**
  with the RISC combinatorics package GeneratingFunctions
  [Christian Mallinger, ``Algorithmic Manipulations and Transformations of Univariate Holonomic Functions and Sequences'', Diplomarbeit, RISC-Linz, 1996]

```
In[7]:= GuessNext2Values[Li_] := Module[{rec}, rec = GuessRE[Li, c[k], {1, 2}, {0, 3}];
          RE2L[rec[[1]], c[k], Length[Li] + 1]]

In[8]:= GuessNext2Values[{1, 2, 4, 8, 16}]

Out[8]= {1, 2, 4, 8, 16, 32, 64}

In[9]:= GuessNext2Values[{1, 3, 6, 10, 15, 21}]

Out[9]= {1, 3, 6, 10, 15, 21, 28, 36}

In[10]:= GuessNext2Values[{1, 1, 2, 6, 24, 120}]

Out[10]= {1, 1, 2, 6, 24, 120, 720, 5040}

In[11]:= GuessNext2Values[{1, 1, 2, 3, 5, 8, 13, 21}]

Out[11]= {1, 1, 2, 3, 5, 8, 13, 21, 34, 55}
```

Book Solution

“34. (Leicht. Jede Zahl wird durch Subtraktion der folgenden von der naechstfolgenden Zahl gebildet: 2-1 = 1 usw. bis 34-21 = 13, also ist die fehlende Zahl 34.

Schwierig. Das Quadrat jeder Zahl unterscheidet sich um 1 vom Produkt der Zahlen rechts und links von ihr: $1^2 = 1$, $2\times 1 = 2$; $2^2 = 4$, $1\times 3 = 3$; usw. bis $21^2 = 441$, $13\times 34 = 442$.)"

```
{1, 1, 2, 3, 5, 8, 13, 21, 34}
```

Book Solution

“34. (Leicht. Jede Zahl wird durch Subtraktion der folgenden von der naechstfolgenden Zahl gebildet: 2-1 = 1 usw. bis 34-21 = 13, also ist die fehlende Zahl 34.

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```
{1, 1, 2, 3, 5, 8, 13, 21, 34}
```

Note. Die “schwierige” Lösung entspricht der Cassini (1680) Identität.

$$F[n+1]\,F[n-1] - F[n]^2 = (-1)^{n+1}, \qquad n \geqslant 1.$$

We shall see, this can be proved automatically with the GeneratingFunctions package!

Idea behind Automatic Guessing

```
? GuessRE
```

```
In[12]:= GuessRE[{1, 2, 4, 8, 16}, c[k]]
Out[12]= {{-2 c[k] + c[1 + k] == 0, c[0] == 1}, ogf}

In[13]:= GuessRE[{1, 3, 6, 10, 15, 21}, c[k]]
Out[13]= {{(-3 - k) c[k] + (1 + k) c[1 + k] == 0, c[0] == 1}, ogf}

In[14]:= GuessRE[{1, 1, 2, 6, 24, 120}, c[k]]
Out[14]= {{(-1 - k) c[k] + c[1 + k] == 0, c[0] == 1}, ogf}

In[15]:= GuessRE[{1, 1, 2, 3, 5, 8}, F[k]]
Out[15]= {{-F[k] - F[1 + k] + F[2 + k] == 0, F[0] == 1, F[1] == 1}, ogf}
```

Application: QuickSort

Recurrences : guessing, etc.

We start with the recursive description:

In[16]:= `{F[0] = 0, F[1] = 0}`

Out[16]= {0, 0}

In[17]:= $F[n\_] := \frac{1}{n}\left((n-1)\,n + 2\sum_{k=0}^{n-1} F[k]\right)$

In[18]:= `upto11 = Table[F[n], {n, 0, 11}]`

Out[18]= $\left\{0, 0, 1, \frac{8}{3}, \frac{29}{6}, \frac{37}{5}, \frac{103}{10}, \frac{472}{35}, \frac{2369}{140}, \frac{2593}{126}, \frac{30\,791}{1260}, \frac{32\,891}{1155}\right\}$

Let's guess the next two values:

In[19]:= `GuessNext2Values[upto11]`

Out[19]= $\left\{0, 0, 1, \frac{8}{3}, \frac{29}{6}, \frac{37}{5}, \frac{103}{10}, \frac{472}{35}, \frac{2369}{140}, \frac{2593}{126}, \frac{30\,791}{1260}, \frac{32\,891}{1155}, \frac{452\,993}{13\,860}, \frac{476\,753}{12\,870}\right\}$

Recall: we guessed the next two values:

In[20]:= `GuessNext2Values[upto11]`

Out[20]= $\left\{0, 0, 1, \frac{8}{3}, \frac{29}{6}, \frac{37}{5}, \frac{103}{10}, \frac{472}{35}, \frac{2369}{140}, \frac{2593}{126}, \frac{30791}{1260}, \frac{32891}{1155}, \frac{452993}{13860}, \frac{476753}{12870}\right\}$

Let' s check :

In[21]:= `upto13 = Table[F[n], {n, 0, 13}]`

Out[21]= $\left\{0, 0, 1, \frac{8}{3}, \frac{29}{6}, \frac{37}{5}, \frac{103}{10}, \frac{472}{35}, \frac{2369}{140}, \frac{2593}{126}, \frac{30791}{1260}, \frac{32891}{1155}, \frac{452993}{13860}, \frac{476753}{12870}\right\}$

Next we guess a recurrence :

In[22]:= `GuessRE[upto11, a[n]]`

Out[22]= $\{\{(1+n)\ (2+n)\ a[n] + \left(-1-5\,n-2\,n^2\right)\ a[1+n] + n\ (2+n)\ a[2+n] == 0,$
$a[0] == 0, a[1] == 0, a[2] == 1\}, \text{ogf}\}$

In[23]:= `GuessedRec = %[[1]]`

Out[23]= $\{(1+n)\ (2+n)\ a[n] + \left(-1-5\,n-2\,n^2\right)\ a[1+n] + n\ (2+n)\ a[2+n] == 0,$
$a[0] == 0, a[1] == 0, a[2] == 1\}$

We guessed the following recurrence for the QuickSort numbers:

In[24]:=
```
GuessedRec
```

Out[24]= $\{(1+n)\ (2+n)\ a[n] + (-1-5n-2n^2)\ a[1+n] + n\ (2+n)\ a[2+n] == 0,\ a[0] == 0,\ a[1] == 0,\ a[2] == 1\}$

In[25]:=
```
? RE2L
```

In[26]:=
```
RE2L[GuessedRec, a[n], 13]
```

Out[26]= $\left\{0, 0, 1, \frac{8}{3}, \frac{29}{6}, \frac{37}{5}, \frac{103}{10}, \frac{472}{35}, \frac{2369}{140}, \frac{2593}{126}, \frac{30791}{1260}, \frac{32891}{1155}, \frac{452993}{13860}, \frac{476753}{12870}\right\}$

Let' s check :

In[27]:=
```
upto13 = Table[F[n], {n, 0, 13}]
```

Out[27]= $\left\{0, 0, 1, \frac{8}{3}, \frac{29}{6}, \frac{37}{5}, \frac{103}{10}, \frac{472}{35}, \frac{2369}{140}, \frac{2593}{126}, \frac{30791}{1260}, \frac{32891}{1155}, \frac{452993}{13860}, \frac{476753}{12870}\right\}$

Another recurrence for the QuickSort numbers:

In[28]:=
```
rec = {(n + 3) a[n + 3] - (3 n + 8) a[n + 2] + (3 n + 7) a[n + 1] - (n + 2) a[n] == 0,
   a[0] == 0, a[1] == 0, a[2] == 1};
```

In[29]:=
```
RE2L[rec, a[n], 13]
```

Out[29]= $\left\{0, 0, 1, \frac{8}{3}, \frac{29}{6}, \frac{37}{5}, \frac{103}{10}, \frac{472}{35}, \frac{2369}{140}, \frac{2593}{126}, \frac{30791}{1260}, \frac{32891}{1155}, \frac{452993}{13860}, \frac{476753}{12870}\right\}$

Let' s check :

In[30]:=
```
upto13 = Table[F[n], {n, 0, 13}]
```

Out[30]= $\left\{0, 0, 1, \frac{8}{3}, \frac{29}{6}, \frac{37}{5}, \frac{103}{10}, \frac{472}{35}, \frac{2369}{140}, \frac{2593}{126}, \frac{30791}{1260}, \frac{32891}{1155}, \frac{452993}{13860}, \frac{476753}{12870}\right\}$

This confirms that this is indeed another recursive description of the QuickSort numbers!

Differential Equations from Recurrences

So far we have not proved that our guessed recurrence

In[31]:=
```
GuessedRec
```

Out[31]= $\{(1+n)\ (2+n)\ a[n] + (-1-5n-2n^2)\ a[1+n] + n\ (2+n)\ a[2+n] == 0,$
$a[0] == 0, a[1] == 0, a[2] == 1\}$

is indeed a valid recurrence for the QuickSort numbers.

We will prove this by transforming the GuessedRec into a DIFFERENTIAL EQUATION for the generating function

$$f(x) = \sum_{n=0}^{\infty} F(n)\, x^n = \sum_{n=0}^{\infty} a[n]\, x^n = -\frac{2x}{(1-x)^2} - \frac{2}{(1-x)^2} \log(1-x)$$

In[32]:=
```
GuessedDE = RE2DE[GuessedRec, a[n], f[x]]
```

Out[32]= $\{2\ (1+x)\ f[x] + (-1-3x+4x^2)\ f'[x] + (x-2x^2+x^3)\ f''[x] == 0, f[0] == 0, f'[0] == 0\}$

Recall: we guessed the DE

In[33]:=
```
GuessedDE
```

Out[33]= $\{2\ (1+x)\ f[x] + (-1-3x+4x^2)\ f'[x] + (x-2x^2+x^3)\ f''[x] == 0, f[0] == 0, f'[0] == 0\}$

We let Mathematica solve it :

In[34]:=
```
DSolve[GuessedDE, f[x], x]
```

DSolve::bvsing :
Unable to resolve some of the arbitrary constants in the general solution using the given boundary conditions. It is possible that some of the conditions have been specified at a singular point for the equation. ≫

Out[34]= $\left\{\left\{f[x] \to \frac{C[1]\ (\pi + i\,x + i\,\mathrm{Log}[-1+x])}{\pi\ (-1+x)^2}\right\}\right\}$

Let' s try the same with our second recursive description :

In[35]:=
```
rec
```

Out[35]=
$$\{-(2+n)\,a[n]+(7+3n)\,a[1+n]-(8+3n)\,a[2+n]+(3+n)\,a[3+n]=0,\ a[0]=0,\ a[1]=0,\ a[2]=1\}$$

Let' s try the same with our second recursive description :

In[36]:=
```
recDE = RE2DE[rec, a[n], f[x]]
```

Out[36]=
$$\left\{-2x-2\left(1-2x+x^2\right)f[x]-\left(-1+3x-3x^2+x^3\right)f'[x]=0,\ f[0]=0\right\}$$

NOTE. This is the same differential equation as derived in the lecture!

In[37]:=
```
DSolve[recDE, f[x], x]
```

Out[37]=
$$\left\{\left\{f[x]\to\frac{2\,i\,(\pi+i\,x+i\,\mathrm{Log}[-1+x])}{(-1+x)^2}\right\}\right\}$$

In[38]:=
$$\mathrm{FullSimplify}\left[\frac{2\,i\,(\pi+i\,x+i\,\mathrm{Log}[-1+x])}{(-1+x)^2}-\left(-\frac{2x}{(1-x)^2}-\frac{2}{(1-x)^2}\mathrm{Log}[1-x]\right)\right]$$

Out[38]=
$$\frac{2\,(i\,\pi+\mathrm{Log}[1-x]-\mathrm{Log}[-1+x])}{(-1+x)^2}$$

In[39]:=
$$\frac{2\,(i\,\pi+\mathrm{Log}[1-x]-\mathrm{Log}[-1+x])}{(-1+x)^2}\ /.\ x\to-0.7$$

Out[39]= $0.+0.\,i$

Consider the sum

In[52]:= $\sum_{k=0}^{n-2} k\, k!$

Out[52]= $-1 + (-1+n)!$

For which values of n is this answer correct?

Alternatively, we can use the procedure from the RISC package `fastZeil`:

In[66]:=
```
Unprotect[Show]; Gosper[k k!, {k, 0, n - 2}]
```

```
If `-2 + n' is a natural number, then:
```

Out[66]=
```
{Sum[k k!, {k, 0, -2 + n}] == -1 + (-1 + n) (-2 + n)!}
```

Note that -in contrast to *Mathematica*- the output of `Gosper` includes additional information about the domain of admissible n!

Execute

In[67]:=
```
Prove[]
```

Explain why this is delivering a proof of the `Gosper output'.

An Automatic Proof of the Binomial Theorem

We prove the binomial theorem in the form

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

Set

In[62]:= $\mathbf{g[n\_] := \sum_{k=0}^{n} \binom{n}{k}_* x^k y^{n-k}}$

In[63]:=
```
{g[0], g[1], g[2], g[3], g[4], g[5], g[6], g[7], g[8]} // Factor
```

Out[63]= $\{1, x+y, (x+y)^2, (x+y)^3, (x+y)^4, (x+y)^5, (x+y)^6, (x+y)^7, (x+y)^8\}$

Proof.

In[64]:= Unprotect[Show]; Zb$\left[\binom{n}{k}_* x^k y^{n-k}, \{k, 0, n\}, n, 1\right]$

```
If `n' is a natural number, then:
```

Out[64]= $\{(x+y)\,\mathrm{SUM}[n] - \mathrm{SUM}[1+n] == 0\}$

In[65]:=
```
Prove[]
```